Damage-spreading dynamic scaling for the Ising model on the Sierpinski gasket fractal

C. Argolo,[1,2] A. Mariz,[1] M. Lyra,[3] and S. Miyazima[4]
[1]*Departamento de Física, Centro Federal de Educação Tecnológica de Alagoas, 57000-000 Maceió, AL, Brazil*
[2]*Departamento de Física Universidade Federal do Rio Grande do Norte, 59072-970 Natal, RN, Brazil*
[3] *Departamento de Física, Universidade Federal de Alagoas, 57072-970, Maceió, AL, Brazil*
[4]*Department of Engineering Physics, Chubu University, Kasugai, Aichi 487, Japan*
(Received 20 July 1999)

We study the relaxation towards equilibrium of the ferromagnetic Ising model on the Sierpinski gasket, which is a fractal lattice. We do this by performing Monte Carlo simulations, based on the heat-bath dynamics, and investigating the time evolution of the Hamming distance between two different configurations of the model. Starting with an initial damage created in all lattice sites, we calculate the average values of two quantities that characterize the relaxation process: the nonlinear damage relaxation time (τ), and the time for all sites to be undamaged at least once (τ_c). We find that τ diverges, at low temperatures, with a dynamical exponent z which depends linearly on the inverse of temperature, as predicted by a generalized scaling theory developed by Henley. There is a complete breakdown of scaling for τ_c.

PACS number(s): 64.60.Ak, 64.60.Ht, 05.10.−a

I. INTRODUCTION

In the numerical simulation of model systems, one main question concerns the relaxation time (τ) necessary to drive the system to a set of microscopic configurations characterizing the thermodynamical equilibrium. This time τ is related to the correlation length (ξ) through $\tau \sim \xi^z$, where z is a dynamic scaling exponent [1]. According to the standard scaling theory, we expect z to be a function of several aspects of the model, such as symmetries and lattice dimensionality but insensible to temperature changes.

However, experimental results on percolation clusters [2] seem to suggest that for models defined on fractal substrates with a vanishing critical temperature, z could also be temperature dependent. This has stimulated further developments (such as generalized scaling theory [3,4]), which predicts that for low temperatures $z \propto T^{-1}$, due to a logarithmic size dependence of energy barriers. However, a renormalization group analysis [5] found z to be independent of T. The dynamical scaling behavior of model systems defined on fractal substrates of zero critical temperature has been investigated [6,7] in the Ising model defined on the Sierpinski gasket, where the thermodynamic behavior can be exactly obtained [8]. This model presents interesting nontrivial physical properties, mainly those concerned with its extrapolation to the thermodynamic limit. Monte Carlo simulations results seem to confirm the generalized scaling theory for thermodynamic quantities with some reported quantitative deviations from the theoretically predicted coefficients [7].

Recently, there has been a renewed interest in the study of the dynamical behavior of model systems defined on fractal objects. The concept of self-organized criticality has provided a link between the widespread occurrence in nature of fractal structures and the phenomenon of $1/f$ noise [9]. Several dynamical properties of fractal systems such as the random-walk scaling laws, reaction kinetics, and vibrational excitations, among others, have been shown to depict new features not present on regular Euclidean lattices [10–12].

Here, we study some important aspects of spin dynamics on fractal systems related to the damage-spreading analysis (see, for example, Refs. [13–24]). This method consists of monitoring the simultaneous time evolution of two initially different microscopic configurations of the model and measuring the fraction of corresponding sites where the spin variables are in different states. This fraction defines the total damage. It is known that several model systems present a dynamic phase transition, separating a chaotic region (where the damage remains finite) and a frozen one (where the damage heals). Here, we will be particularly interested in studying the scaling behavior of the damage relaxation and damage covering times and exploring their relationship with the generalized scaling theory. Our numerical results show that, although the damage relaxation time follows the generalized dynamic scaling, the damage covering time depicts a logarithmic size dependence and therefore violates the dynamical scaling.

II. GENERALIZED SCALING FOR FRACTAL LATTICES

The generalized scaling, as applied to the Ising model on a fractal lattice, is based on the fact that the relaxation time $\tau(L,T)$ of a system of linear size L at a temperature T is determined by an Arrhenius law

$$\tau(L,T) = \tau_0 \exp[\Delta E(L)/k_B T], \tag{1}$$

where τ_0 is a time scale constant and $\Delta E(L)$ is the minimal energy barrier that the system needs to flip from all spins up to all spins down [3,4,6,7]. Generalized scaling assumes that a local quantity like the magnetization M has to be dynamically scaled as

$$M(L,T,t) = M(L/b, T', t/\Omega). \tag{2}$$

Therefore, we have to rescale the time scale by a factor Ω when the size of the lattice is rescaled by a factor b. The renormalized temperature T' can be determined by a scaling transformation procedure [8].

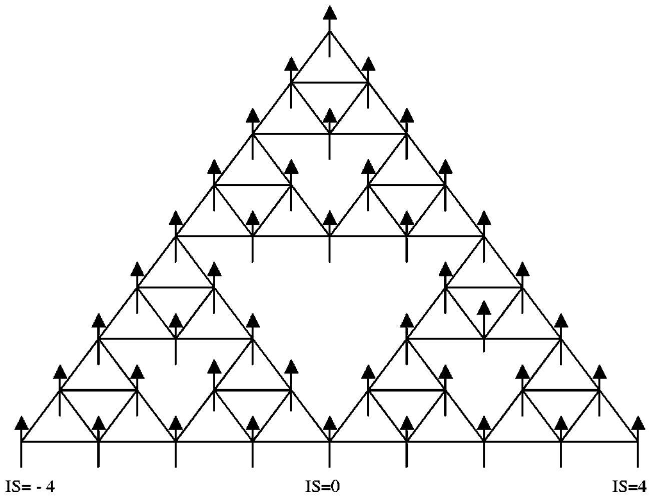

FIG. 1. Sierpinski gasket with four generations. The baseline sites are numbered in such a way that the central one has site index IS=0.

Henley showed [3,4] that the characteristic energy difference ΔE is given by

$$\Delta E(L)/2J \approx Z \ln L + C, \tag{3}$$

where Z is a constant that characterizes the fractal. Then putting Eq. (3) into Eq. (1), the leading behavior for low temperatures ($L \ll \xi$) becomes $\tau(L,T) \approx \tau_0 L^{2ZK}$, where $K = J/k_B T$ is the coupling constant. To reproduce the above predicted behavior, the generalized scaling requires the magnetization to present the dynamic scaling form in Eq. (3), with the rescaling of the relaxation time satisfying

$$\tau(L,K) = \Omega(b,K)\,\tau(L/b,K'). \tag{4}$$

Note that the temperature dependence of the rescaling factor Ω characterizes the basic difference between ordinary and generalized dynamic scalings. Further, to reproduce the correct low temperature behavior, the following ansatz $\Omega = b^{f(K)}$ is assumed, where the function $f(K)$ has the simple low-temperature form

$$f(K) = a_0 K + a_1 + O(1/K). \tag{5}$$

Taking $b = L$ it is obtained that

$$\tau(L,K) = \Omega(L,K)\,\tau'(1,K'), \tag{6}$$

resulting in the leading behavior $\tau(L,K) \approx L^{a_0 K}$. Hence, the linear coefficient of $f(k)$ is related to the geometric factor Z by $a_0 = 2Z$.

To calculate the value of a_0, we measure $\Omega = \tau/\tau'$, considering systems of different sizes and temperatures. $f(K)$ is obtained by making the best linear fit to the plot of $\log_b \Omega$ versus K, in the limit of $K \to \infty$.

III. MODEL AND FORMALISM

The Hamiltonian of the Ising ferromagnet model with nearest-neighbor interactions is

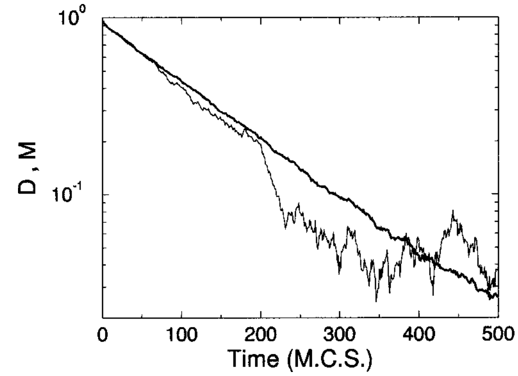

FIG. 2. Semilogarithmic plot of the *damage* (thicker line) and *magnetization* vs time (in MCS). Notice that the signal of the damage is more stable than that of the *magnetization*. In this case, the damage is more reliable for statistics due to fewer fluctuations.

$$H = -J \sum_{\langle i,j \rangle} S_i S_j , \tag{7}$$

where $J > 0$ and $\langle i,j \rangle$ denotes nearest-neighbor sites on the Sierpinski gasket, as illustrated in Fig. 1. On this lattice, the geometric factor $Z = 2/\ln 2$ and $C = 4$. The thermodynamic properties of this system can be obtained numerically by implementing a standard Monte Carlo algorithm with specific dynamical rules. In what follows, we will investigate, through a damage-spreading technique, some aspects related to the relaxation to equilibrium when it is driven by a heat-bath dynamics. The damage technique is introduced as follows: we take two system configurations A and B which differ from each other by a given set of spins which are put in distinct states. The simultaneous temporal evolution of the two copies is performed using the same Monte Carlo rules and the same random numbers (for the same sites on lattices A and B). The Hamming distance is a measure of the damage and is defined as the fraction of sites which have different spin orientations on system configurations A and B, i.e.,

$$D(t) = \frac{1}{2N} \left\langle \sum_i \; |S_i^A(t) - S_i^B(t)| \right\rangle, \tag{8}$$

where N is the total number of sites and the brackets stand for an average over many samples (different initializations of the random number generator).

The present Monte Carlo (MC) simulation has started with lattices of size L (ranged from $L=4$ to $L=128$) at some coupling K with all spins up in copy A and down in copy B, corresponding to an initial damage $D(0)=1$. We recorded the damage per site as a function of time t, where each Monte Carlo step per spin (MCS) represents a unit of time. We have also averaged over until 10^4 experiments, using free boundary conditions in order to avoid the breakdown of hierarchy of the lattice. As this model is paramagnetic at any finite temperature, the asymptotic equilibrium value of the damage is zero (frozen phase). We investigate its relaxation properties by calculating two quantities: the nonlinear damage relaxation time [$\tau = \int_0^\infty D(t)\,dt$], and the time for all sites to be undamaged at least once (τ_c). It should be stressed that the damage-spreading technique has been successfully used

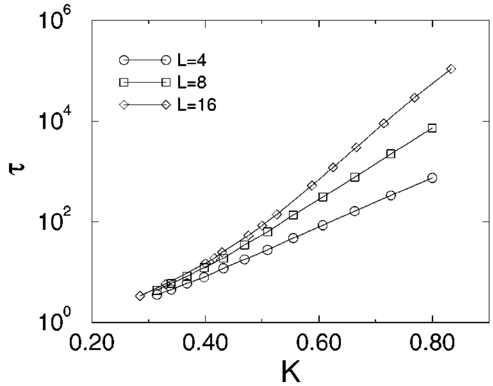

FIG. 3. Semilogarithmic plot of the relaxation time vs inverse temperature for three different values of $L(L=4,8,16)$. The curves show that the slope of log τ vs K depends on L for low temperatures. In this case τ is a power law of the form $\tau \sim L^{a_0 K}$.

to determine the dynamical critical exponent z of the two- and three-dimensional Ising model [22,25].

IV. RESULTS

For simplicity, in all our results we will consider $J=k_B=1$. In Fig. 2, we plot the time evolution of the damage and the magnetization. This figure shows that the magnetization fluctuations are larger when compared with that of the damage, which presents a much smoother behavior making it possible to get reasonably good statistical results. Besides, in contrast with the magnetization, the damage becomes zero if we wait enough time, and remains null. We recall that, as the system presents no phase transition (except for $T=0$ and $L\to\infty$), only an exponential decay is observed.

Figure 3 shows the relaxation time τ as a function of K (inverse temperature) for three different values of L. These curves show that the slopes of log τ versus K slowly grows with L for low temperatures, confirming the predicted trend. For high temperatures (low K) the slopes appear to be the same. Figure 4 shows the relaxation time τ as a function of L for different temperatures.For low temperatures ($L \ll \xi$) the data is consistent with the assumed dominant power-law behavior $\tau \simeq L^z$ with a temperature dependent exponent. When L is increased, a downward curvature appears with τ saturating for large L as expected in the limit $L \gg \xi$, where $\tau \propto \xi^z$.

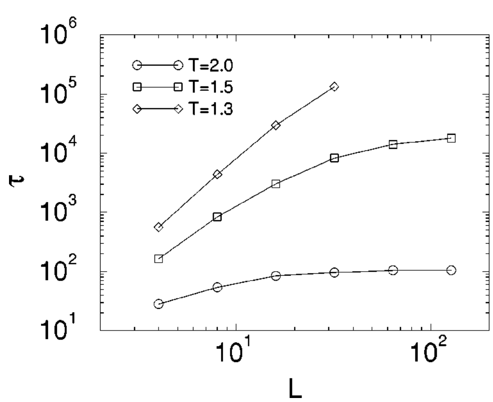

FIG. 4. Relaxation time τ vs L for different values of T. We observe two regimes: a high temperature one, where τ saturates, and a low temperature one, where the curve seems to agree with a power law.

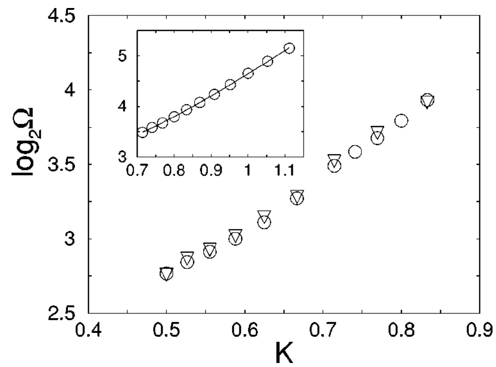

FIG. 5. Plot of $\log_2\Omega$ vs K from $L=8$ to $L'=4$ (circles) and from $L=16$ to $L'=8$ (squares) renormalizations. The solid line is a fitting for the nonlinear function $f(K)=a_0K+a_1+a_2/K$, where $a_0=2Z$. The best fit gives the values of $a_0=5.7\pm0.1$, $a_1=-2.2\pm0.2$ and $a_2=1.1\pm0.1$.

After calculating τ for a system of size L at a coupling K the same procedure is repeated now for τ' in a system of size $L'=L/b$ but at a coupling K', calculated by a renormalization procedure with $b=2$ [8] given by

$$\exp(4K')=\frac{\exp(8K)-\exp(4K)+4}{\exp(4K)+3}. \qquad (9)$$

The renormalization factor Ω is calculated by the relation $\Omega=\tau/\tau'$, where τ and τ' are the relaxation times in each of the correspondent systems. A plot of $\log_b \Omega$ versus K appears in Fig. 5 for renormalizations of systems with $N=42$ ($L=8$) to $N'=15$ spins ($L'=4$) and with $N=123$ ($L=16$) to $N'=42$ spins ($L'=8$). The data collapse and the temperature dependence of $\log_2\Omega$ reinforces the validity of the generalized scaling hypothesis. This figure is to be compared with Fig. 5 of Ref. [7]. Looking closely, our result suggests that in the range of temperatures investigated, a $1/K$ term in the $f(K)$ expansion seems to be a relevant correction to the leading linear behavior. We fit the data to the expression

$$f(K)=a_0K+a_1+a_2/K, \qquad (10)$$

finding that the best nonlinear fit gives a value of $a_0=5.7\pm0.1$, which is in better agreement with Henley's result

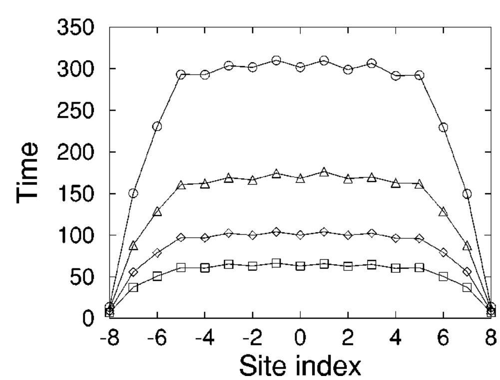

FIG. 6. Average time (in MCS) for each site of lattice base line ($L=16$) to be undamaged (healed). From bottom to top: $T=1.5$, 1.4, 1.3, and 1.2.

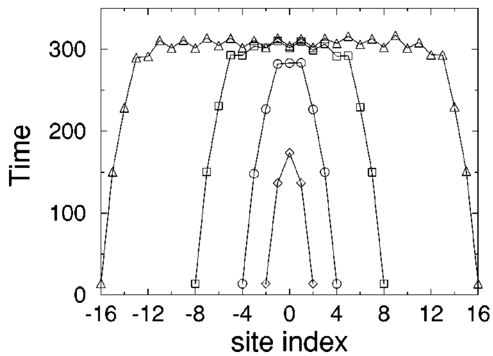

FIG. 7. Average time (in MCS) for each site of the lattice base line ($L=4,8,16,32$ from the inside to the outside) to be undamaged (healed). Curves have been calculated at $T=1.2$ and have been averaged over 10^4 samples.

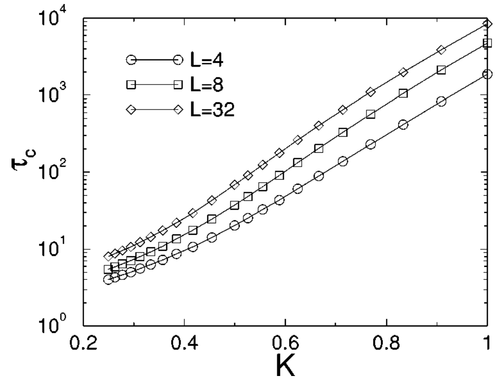

FIG. 9. Semilogarithmic plot of the covering time against inverse temperature for three different values of $L(L=4,8,32)$. These curves show that the slope of log τ_c vs K does not depend on L.

$(a_0)_H=4/\ln 2=5.77$, than the former one $a_0=4.2$ obtained from a crude linear fit in the same temperature range [7].

To have more insight into the system dynamics we have also examined another characteristic time of evolution of the damage (τ_c) defined as the time for all sites to be undamaged at least once [26]. To investigate the scaling properties of τ_c, we proceed in the same way done for τ calculating now Ω_c from τ_c and τ_c'.

Figure 6 shows the average time (in MCS) for each site of the lattice base line to be undamaged (or healed) at least once (for $L=16$). We notice that, when the temperature is decreased, we have at least four well distinguishable characteristic time scales. These curves have been averaged for 10^4 samples. The corners of the gasket very soon become undamaged, as expected, since the sites at the corners are less stable to fluctuations. From the fourth spin on, in the direction to the center of the gasket (from the left or from the right), all spins become undamaged at the same time scale. In Fig. 7 we show the τ_c base line profile for distinct sizes. It shows that the existence of four time scales persists for any L with the inner sites $|i|\leqslant L/2-3$ being undamaged almost at the same time despite a small regular oscillation. The presence of distinct characteristic times scales is a signature of a possible scaling violation.

Figure 8 shows the total covering time $\tau_c/\ln L$ versus L. All curves saturate on a constant value showing that there is only one regime with τ_c growing with $\ln L$ for all T. The semilog plot of τ_c against the inverse of temperature is shown in Fig. 9. From these curves, we observe that the slope of log τ_c versus K is not L dependent contrary to the behavior of τ (see Fig. 3). The results in Figs. 8 and 9 indicate that τ_c is not a power law, as observed for the relaxation time τ in the low temperature regime. The logarithmic size dependence of the damage covering time is similar to the one observed to hold for the energy barriers.

Figure 10 shows the logarithm plot of Ω_c versus K for three different renormalizations ($L/L'=32/16$, 16/8, and 8/4). In contrast with the results obtained from the relaxation time, the scaling factor of the damage lattice covering time τ_c appears to depend not only on K and b but also on the proper system size L. In other words, τ_c does not satisfy the generalized dynamic scaling, specially in the low temperature regime where the difference of the characteristic damage covering time scales is more proeminent. It is interesting to notice that all curves show that there is a crossover around $T=2$. This temperature is close to a characteristic temperature of the system that has been reported some time ago [27], which relates to the maximum of the specific heat of this system (Schottky peak anomaly) occurring at $T\cong 2J/K$. Similar results [28] were found for the Ising model on the three-dimensional Sierpinski gasket with $T\cong 3J/K$.

V. CONCLUSION

We have studied the spin dynamics on a fractal lattice (the Sierpinski gasket) through a Monte Carlo damage spreading

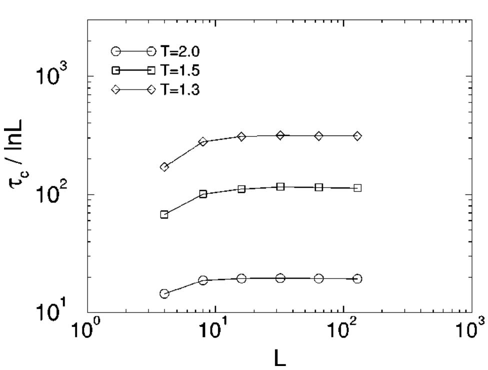

FIG. 8. Log-log plot of ($\tau_c/\log L$) vs L. They saturate as $T\to\infty$ showing that τ_c grows with $\log L$ for all T.

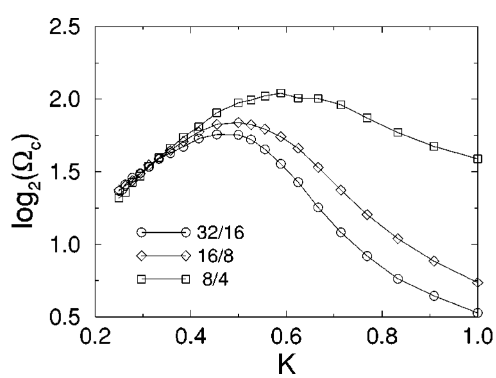

FIG. 10. Semilogarithm plot of $\Omega_c=\tau_c/\tau_c'$ vs K for three different renormalizations ($L/L'=32/16$, 16/8, and 8/4). The system shows no scaling of τ_c especially for low temperatures. All curves show that there is a crossover around $T=2$.

technique using heat-bath dynamics. The damage relaxation time was found to follow a generalized dynamic scaling as proposed by Henley [3,4]. Accurate data for the renormalization scaling factor of the relaxation time provide a precise estimate of $z=a_0K$, with $a_0=5.7\pm0.1$ in agreement with the scaling prediction and substantially better than previous Monte Carlo results. Further, we also investigated the total damage covering time, which was found not to obey a scaling form. We conjecture that the breakdown of dynamic scaling of the covering time is a universal characteristic of scale invariant systems. At present we are investigating the validity of this conjecture on fractal lattices that present a finite critical temperature for which a standard dynamic scaling of the relaxation time is expected to hold.

ACKNOWLEDGMENTS

We acknowledge Heber R. da Cruz and G.M. Viswanathan for very useful discussions. We also thank CAPES and CNPq for financial support.

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